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MODERN APPROACHES TO THE FORMATION OF METHODOLOGICAL COMPETENCE FUTURE TEACHERS

Key words: competences, competence approach, content of education, postgraduate education, andragogy

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VORNICU-SCHUR THEOREMS AND METHODOLOGY AND PRACTICE OF PROVING SOME TYPES OF SYMMETRIC HOMOGENEOUS OLYMPIAD INEQUALITIES

Key words: professional teacher development, mathematical and methodological competence, mathematics teaching methodology, mathematically gifted students, olympiad-type problems in mathematics, methods for proving symmetric inequalities

The world and domestic experience of working with mathematically gifted students demonstrates the significance of the inequalities proving theory as one of the most important aspects of classical olympiad preparation at all levels. The systematization,

generalization, and adaptation of mathematical and didactic-methodical resources in this field should be an influential component of teacher development programs in mathematics, aimed at creating convoluted didactic structures by means of specialization and transformation of high-level theoretical material [2].

The National Doctrine of Education Development, approved by the Order of the President of Ukraine dated April 17, 2002, No. 347/2002, emphasizes the combination of education and science as a obligatory condition for modernizing the education system based on its fundamentalization. The fundamentalization of education has been declared one of the priorities of the Bologna Process (UNESCO Special Memorandum, 1994). Therefore, as shown in research by S. U. Honcharenko, M. M. Kovtoniuk, and others, the principle of educational fundamentalization should be considered as a universal didactic category that underlies modern multi-level education. The Department of Teaching Methodology and Educational Content of the Odessa Academy of In-Service Education introduces the paradigm of the integrity of subject-methodological competencies of teachers during their professional development through the modeling of theoretical and practical cases in collective and collectively distributed creativity [3]. We believe that the approaches to systematizing and selecting educational content in the context of fundamentalization processes in profile (specialized) high school and higher education institutions, analyzed by N. V. Zakharchuk [1], are relevant not only for postgraduate (continuing) pedagogical education but also for the professional development of teachers from the standpoint of their knowledge culture and the progressive expansion of scientific knowledge acquired during higher education.

The mathematical and didactic-methodological support with preparation of students for mathematical competitions (including teacher professional development) should fully rely on the principles of fundamentalization because the content of olympiads is often generated by the creative work of mathematicians-researchers who transform scientific narratives (results) into olympiad problems. Teaching the solution of difficult olympiad problems is inevitably integrated with the constant updating of scientific knowledge of all participants of the educational process. For this reason, there are complex issues related to the concentration of material at relevant levels, preventing the unjustified reduction in the quality of its presentation (for example, proving new theorems for teachers and students involves important ideas related to

creating and solving problems, studying the combination of different mathematical techniques, and thereby, the analysis of proofs of theoretical facts is a fundamental aspect of olympiad preparation). It is necessary to constantly monitor new publications that can have a real impact on the emergence of new types of problems, new methods of solving known problems etc. For olympiad problems, regardless of their didactic typology (reproductive, heuristic, exploratory, creative), it is relevant to consider *taxonomic approaches* to the hierarchy of goals, processes, and outcomes in the cognitive domain.

The methodological effectiveness and quality of consolidating educational outcomes, both in the education of gifted students and in working with teachers, are significantly increased if the isolation and coordination of mathematical and didactic connections occur through the gradual motivated movement of task-theoretical chains and / or blocks, which provide the opportunity to discuss various methods of solving for most tasks, analyze, synthesize, and reflexively evaluate, forming and synchronously filling *aspects of the schoolchildren's nearest mathematical development* and corresponding aspects of the teacher's professional development [4–6].

We emphasize that in training students with advanced mathematical educational needs and in the professional development of mathematics teachers, the theory of inequalities has proven to be one of the most open areas for implementing the principles of fundamentalization of education. Proving inequalities involves a diversified arsenal of well-known (model) inequalities, special theorems, and problems that arise during mathematical research. Proving inequalities uses quite non-standard algebraic techniques, the conceptual and operational tools of mathematical analysis, and many inequalities require a geometric interpretation. Teaching the proof of inequalities, even at the goal-setting stage, successfully combines with the stimulation of mastering a new mathematical environment [4].

Let's consider the ways of methodological modeling of teaching one of the methods of proving symmetric homogeneous inequalities.

In many papers and textbooks, examples of using *the Inequality of Three Squares* ($a^2 + b^2 + c^2 \geq ab + bc + ca$; a, b, c are arbitrary real numbers) are provided for proving inequalities. This classic inequality can be proven in various ways, including some that are quite «elementary» and others that require more advanced (olympiad-level) preparation (particularly, using the Cauchy-Bunyakovsky Inequality, the Rearrangement Inequality, and examining properties of the quadratic

function $f(a) = a^2 - a(b + c) + (b^2 + c^2 - bc)$, etc.). This is intended to form *the first block* of acquiring new competencies in the chosen subject. At the same time, the symmetric Inequality of Three Squares can be expressed as $(a - b)(a - c) + (b - a)(b - c) + (c - a)(c - b) \geq 0$, which can be simplified to $\sum_{cycl} (a - b)(a - c) \geq 0$. Without loss of generality, we assume that $a \geq b \geq c$, then the proof follows from $(c - a)(c - b) \geq 0$, $(a - b)(a - c) + (b - c)(b - a) = (a - b)^2 \geq 0$. The well-known Schur's theorem (Issai Schur, 1876–1941) states that for $\lambda \geq 0$ and arbitrary non-negative numbers a, b, c the inequality $\sum_{cycl} a^\lambda (a - b)(a - c) \geq 0$ holds (of course, for $\lambda = 0$ we «formally» accept that $u^0 = 1$ for all $u \in R$). Since $ab + bc + ca \leq |a||b| + |b||c| + |c||a|$, it suffices to prove the Inequality of Three Squares only for non-negative numbers, and we can see that it is a special case of Schur's theorem with $\lambda = 0$ (note that in fact the Inequality of Three Squares is sufficient to prove for positive real numbers a, b, c).

The second methodological block is a practical workshop on proving olympiad inequalities using the Schur's inequality, which can be based on publications [7; 8; 10].

Let's consider the expression (cyclic sum) $R_f(a, b, c) = \sum_{cycl} f(a)(a - b)(a - c)$, which, as it is easy to see, is symmetric. In 2003, Valentin Vornicu published elegant and productive generalizations of Schur's theorem with simple proofs (*the third methodological block*) [11; 12].

Theorem 1 (V. Vornicu). Let $f: \langle \alpha; \beta \rangle \rightarrow R_+$ be a non-decreasing in interval $\langle \alpha; \beta \rangle$ function, $-\infty \leq \alpha < \beta \leq +\infty$. Then for any a, b and c from interval $\langle \alpha; \beta \rangle$ the inequality $R_f(a, b, c) \geq 0$ holds.

Theorem 2 (V. Vornicu). Let $f: \langle \alpha; \beta \rangle \rightarrow R_+$ be a convex in interval $\langle \alpha; \beta \rangle$ function, $-\infty \leq \alpha < \beta \leq +\infty$. Then for any a, b and c from interval $\langle \alpha; \beta \rangle$ the inequality $R_f(a, b, c) \geq 0$ holds.

(We use $\langle \alpha; \beta \rangle$ notation, though, the theorems **1** and **2** are valid for all types of intervals, including finite and infinite intervals: $\langle \alpha; \beta \rangle$, $[\alpha; \beta]$, $(\alpha; \beta]$, $[\alpha; \beta)$.)

From Vornicu's theorems it follows that the Schur's inequality holds in various cases, particularly: **i**) a, b and c are arbitrary positive real numbers, λ is arbitrary number; **ii**) a, b and c are arbitrary real numbers, λ is odd integer positive number.

Generalizations of the results of I. Schur and V. Vornicu can be found in the works of mathematicians like D. Grinberg, B. Finta etc. These results can serve as a basis for creating new olympiad problems

and research topics for competitions like the Ukrainian Junior Academy of Sciences (School Science Society). However, the discussion remains within the limits of the facts mentioned above.

When mastering the methodology of proving symmetric inequalities with three variables (considering inequalities that allow the use of the properties of homogeneity) by reducing them to Vornicu's theorems (Vornicu-Schur theorems), there arises a challenge in representing the inequalities in the equivalent form $R_f(a, b, c) \geq 0$ with function f (if it exists) satisfying the conditions of Theorem 1 or Theorem 2 (*fourth methodological block*). Let's illustrate this by a series of educational examples. Various (including formally simpler) methods for solving these problems can be found in specialized and educational literature, so it is worthwhile to begin the discussion of each problem by considering alternative methods of proof.

Example 1 (*Inequality of Three Squares*). Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$, where a, b and c are arbitrary real numbers.

To find the required function f in the relation $a^2 + b^2 + c^2 - ab - bc - ca = \sum_{cycl} f(a)(a-b)(a-c)$ (1.1) we assume $b = c$. Then, for all real numbers a and c the equality $f(a)(a-c)^2 = (a-c)^2$ holds. By considering the following cases: $c = 0$ and $a \neq 0$; $c \neq 0$ and $a = 0$, we obtain $f(a) \equiv 1$, with *necessity*. We verify that for the function $f(x) = 1, x \in R$, the equality (1.1) holds, and this function satisfies the conditions of both Theorem 1 and Theorem 2. ■

Example 2 (*A. Nesbitt*). Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \quad (2.1)$$

for any positive real numbers a, b and c .

The Nesbitt's Inequality (2.1) is a classic *model* problem used to demonstrate different proof techniques to teachers and students. Traditionally, it is shown that this inequality can be simply reduced to the equivalent inequality $\sum_{cycl} (a-b)^2(a+b) \geq 0$, but we want to demonstrate the «power» of Theorems 1 and 2. For the Nesbitt's Inequality we also take into account the property of homogeneity (which is typical) and additionally assume $a + b + c = 1$. We consider the inequality (2.1) and, accordingly, the relation $\sum_{cycl} (a-b)^2(a+b) = \sum_{cycl} f(a)(a-b)(a-c)$ (2.2) on the set $\Omega = \{(a, b, c): a + b + c = 1; a > 0, b > 0, c > 0\}$. Let $b = c$. Then $f(a)(a-b)^2 = 2(a-b)^2(a+b)$, when $a + 2b = 1, a > 0, b > 0$. If $a \in (0; 1) \setminus \left\{\frac{1}{3}\right\}$, then $b \neq a$, and $f(a) = 2a + 2b, f(a) = 2a + 1 - a$, therefore $f(a) =$

$a + 1$ for any $a \in (0; 1) \setminus \{\frac{1}{3}\}$. Let's set $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$ in (2.2) and compute $f(\frac{1}{3}) = \frac{4}{3}$. It now follows that $f(a) = a + 1$ for all $a \in (0; 1)$. We verify that for the function $f(x) = x + 1$, $x \in (0; 1)$, the equality (2.2) holds in Ω (for this, it is convenient to check directly the equality $\sum_{cycl} (a - b)^2(a + b) = \sum_{cycl} (2a + b + c)(a - b)(a - c)$ (2.3) for arbitrary real numbers a, b and c). The function f in the interval $(0; 1)$ satisfies the conditions of both Theorem 1 and Theorem 2. It's worth noting that «seeing» the identity (2.3) to introduce the constraint $a + b + c = 1$ and then applying Vornicu's theorems can be quite challenging. ■

Example 3. Let a, b , and c be real positive numbers. Prove that

$$\frac{a^2+bc}{(b+c)^2} + \frac{b^2+ca}{(c+a)^2} + \frac{c^2+ab}{(a+b)^2} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}. \quad (3.1)$$

Let's rewrite inequality (3.1) in the form $\sum_{cycl} \frac{(a-b)(a-c)}{(b+c)^2} \geq 0$ (3.2). With consideration of homogeneity, inequality (3.1) allows us to introduce the constraint $a + b + c = 1$, after which inequality (3.2) becomes $\sum_{cycl} f(a)(a - b)(a - c) \geq 0$, where $f(t) = \frac{1}{(1-t)^2}$, $t \in (0; 1)$. The function f satisfies the conditions of both Vornicu's theorems. ■

Example 4. Show that if a, b and c are positive real numbers, then

$$\frac{bc(b+c)}{a^2+bc} + \frac{ca(c+a)}{b^2+ca} + \frac{ab(a+b)}{c^2+ab} \geq a + b + c. \quad (4.1)$$

This inequality is equivalent with

$$\frac{abc(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{(a^2+bc)(b^2+ca)(c^2+ab)} \geq 0, \quad (4.2)$$

which obviously follows from the Inequality of Three Squares. However, we are looking for approaches related to Vornicu's theorems.

Let's rewrite inequality (4.1) as $\sum_{cycl} \frac{a(a-b)(a-c)}{a^2+bc} \geq 0$ (4.3).

Note that there is a «risk» of choosing the «wrong» path, leading to what is known as a *false association* [2]. Homogeneity allows us to assume that $abc = 1$, and inequality (4.3) takes the form $R_f(a, b, c) \geq 0$, where

$f(x) = \frac{x^2}{x^3+1}$. But function f increases in the interval $[0; \sqrt[3]{2}]$, decreases

in $[\sqrt[3]{2}; +\infty)$, in the intervals $\left[0; \sqrt[3]{\frac{1}{2}(7 - 3\sqrt{5})}\right]$ and

$\left[\sqrt[3]{\frac{1}{2}(7 + 3\sqrt{5})}; +\infty\right)$ is convex, in the interval

$\left[\sqrt[3]{\frac{1}{2}(7 - 3\sqrt{5})}; \sqrt[3]{\frac{1}{2}(7 + 3\sqrt{5})} \right]$ is concave. Clearly, it is not possible to apply Vornicu's theorems in this way.

Inequality (4.1) can also be expressed in the form

$$\frac{abc}{(a^2+bc)(b^2+ca)(c^2+ab)} \sum_{cycl} (a^2 + ab + bc + ca)(a - b)(a - c) \geq 0.$$

We will use homogeneity in the form $ab + bc + ca = 1$, leading to the inequality $R_f(a, b, c) \geq 0$ for the convex function $f(x) = x^2 + 1$. ■

Other proof methods for examples 3 and 4 are presented in [12].

Example 5 (*Nguyen Duy Tung*). Prove that the inequality

$$a^2 + b^2 + c^2 + abc \sum_{cycl} \frac{b+c}{a^2+bc} \geq 2(ab + bc + ca) \quad (5.1)$$

holds for any positive real numbers a, b and c .

Considering the form of inequality (5.1), it's convenient to use homogeneity with the constraint $abc = 1$. Under these conditions, in $\Omega = \{(a, b, c) : abc = 1; a > 0, b > 0, c > 0\}$ we will attempt to replace inequality (5.1) with an equivalent inequality of the form $R_f(a, b, c) \geq 0$ so that Vornicu's results can be applied. Let $b = c$, so $ab^2 = 1$. If $a \neq b$ (i. e., $a \neq 1$), then on $\{(a, b, c) : abc = 1; a > 0, b > 0, c > 0, b = c, a \neq 1\}$ from equality

$$a^2 + b^2 + c^2 + abc \sum_{cycl} \frac{b+c}{a^2+bc} - 2(ab + bc + ca) = \sum_{cycl} f(a)(a - b)(a - c) \quad (5.2)$$

it follows that for all $a \in (0; +\infty) \setminus \{1\}$ $f(a) = \frac{a^3}{a^3+1}$. For $a = 1, b = 2, c = \frac{1}{2}$ equality (5.2) gives $f(1) = \frac{1}{2}$. Thus, necessarily, $f(x) = \frac{x^3}{x^3+1}$ for all real $x > 0$.

It is convenient to verify by explicitly observing that

$$a^2 + b^2 + c^2 + abc \sum_{cycl} \frac{b+c}{a^2+bc} - 2(ab + bc + ca) - \sum_{cycl} \frac{a^2}{a^2+bc} (a - b)(a - c) = 0$$

for all a, b and c such that $(a^2 + bc)(b^2 + ca)(c^2 + ab) \neq 0$.

The function f increases in the interval $(0; +\infty)$, so we can apply theorem 1, and this completes the proof. ■

Another way to prove inequality (5.1) is provided in [9].

The investigation and systematic implementation of modern approaches to proving inequalities in postgraduate education elevate teachers to a high competence level in the professional realization of mathematical and methodological integration of leading content lines. This creates prerequisites for teachers' search activities, improvement of skills in analyzing materials of high-level mathematical Olympiads regarding connections between problems and new theoretical facts and generalizations of known results. This direction should also be followed when forming the scientific and didactic resource base for future work with mathematically gifted schoolchildren in students of mathematical specialties. The proposed material will be helpful in preparing school students for the All-Ukrainian and International Mathematical olympiads.

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ТЕОРЕМИ ВОРНІКУ-ШУРА ТА МЕТОДИКА Й ПРАКТИКА ДОВЕДЕННЯ ДЕЯКИХ ТИПІВ СИМЕТРИЧНИХ ОДНОРІДНИХ ОЛІМПІАДНИХ НЕРІВНОСТЕЙ

Ключові слова: підвищення кваліфікації вчителів, математична та методична компетентність, методика навчання математики, математично обдаровані учні, олімпіадні задачі з математики, методи доведення симетричних нерівностей